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TECHNICAL NOTES

Effects of natural convection on the inward solidification of spheres and cylinders

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1. INTRODUCTION

THE CLASSICAL theory for the inward solidification of spheres and cylinders has been studied extensively see, for example, Stewartson and Waechter [1]. However in a practical situation the liquid is nearly always at a temperature above fusion temperature. During solidification such temperature differences will induce a non-uniform convective motion and as a consequence the interface will be non-symmetrical. The effects of natural convection in the melted region around cylinders has been investigated by Sparrow *et al.* [2] and [3], Yao and Chen [4] and Ho and Viskanta [5]. In these studies the Stefan number, $\beta = L/C_p^*(T_F - T_0)$, was taken to be large and enabled a quasi-steady approximation to be made for the convective motion and heat transfer.

Here numerical results are given on the effects of natural convection during the solidification of spheres and (horizontal) cylinders. Small time expansions of the governing equations are developed. Such results are relevant to the study of solidification of metal alloys, see Chiesa and Guthrie [6] and Schulze *et al.* [7]. They are also relevant to the study of heat transfer in latent heat-of-fusion storage devices, see Ho and Viskanta [8].

2. ANALYSIS

The following assumptions are made:

(i) there exists a definite fusion temperature at which solidification occurs;
(ii) all thermal and transport properties of the system are independent of the temperature; and
(iii) the density changes at change of phase and elsewhere in the solidification process are ignored except in the calculation of the gravitational buoyancy force within the liquid, i.e. the Boussinesq approximation is invoked.

The physical situation to be examined is illustrated in Fig. 1. Initially the sphere or circular cylinder of radius a is filled with liquid at temperature T_1, above the fusion temperature T_F. At time $t = 0$ the surface $r = a$ is instantaneously reduced to temperature $T_0(< T_F)$ causing the liquid adjacent to the surface to emit latent heat and solidify. The solid/liquid interface moves inwards and is given by

$$r - r_F(\mu, t) = 0, \mu = \cos\theta. \quad (1)$$

The governing equations and boundary conditions are now given.

Liquid phase

Let (v_r^*, v_θ^*) be the velocity components in the (r, θ)-directions and introduce the stream function ψ^* so that

$$v_r^* = -(1-\mu^2)^{(2-m)/2} r^{-m}\, \partial\psi^*/\partial\mu$$

and $\quad (2)$

$$v_\theta^* = -r^{1-m}(1-\mu^2)^{(1-m)/2}\, \partial\psi^*/\partial r,$$

where $m = 1$ and 2 for the cylinder and sphere geometries, respectively. Introduce the dimensionless variables

$$R = r/a, \quad R_F = r_F/a, \quad \tau = kt/a^2, \quad \tau^* = k^*t/a^2$$
$$\theta = (T - T_0)/(T_F - T_0), \quad \theta^* = (T^* - T_F)/(T_1 - T_F) \quad (3)$$

and

$$\Psi^* = \psi^*/k^*a^{m-1},$$

the dimensionless parameters

$$\gamma = k^*(T_1 - T_F)/[k(T_1 - T_0)], \quad \lambda = k^*/k, \quad (4)$$

together with the dimensionless groups: Stefan number $\beta = L/C_p^*(T_F - T_0)$, Prandtl number $P = \nu^*/k^*$ and Rayleigh number $Ra = \alpha^* g(T_1 - T_F)a^3/\nu^* k^*$. Here a * denotes a liquid property; k, the thermal diffusivity; ν, the kinematic viscosity; L, the latent heat; C_p, the specific heat; α, the coefficient of cubical expansion; and g the gravitational constant.

Introduce the following operators:

$$D_R^2 = R^{m-2}\, \partial/\partial R(R^{2-m}\, \partial/\partial R) + R^{-2}(1-\mu^2)^{m/2} \times \partial/\partial\mu[(1-\mu^2)^{(2-m)/2}\, \partial/\partial\mu],$$

$$L_R = [\mu/(1-\mu^2)]\, \partial/\partial R + R^{-1}\, \partial/\partial\mu, \quad (5)$$

$$\nabla_R^2 = R^{-m}\, \partial/\partial R(R^m\, \partial/\partial R) - R^{-2}(1-\mu^2)^{(2-m)/2} \times \partial/\partial\mu[(1-\mu^2)^{m/2}\, \partial/\partial\mu],$$

and the Jacobian

$$\partial(A, B)/\partial(R, \mu) = \partial A/\partial R\, \partial B/\partial\mu - \partial A/\partial\mu\, \partial B/\partial R.$$

The governing equations for the liquid now become

$$D_R^4\Psi = P^{-1}\{\partial D_R^2\Psi^*/\partial\tau^* + R^{-m}(1-\mu^2)^{(2-m)/2} \times [\partial(\Psi^*, D_R^2\Psi^*)/\partial(R, \mu) + 2(m-1)D_R^2\Psi^* L_R\Psi^*]\} + Ra(1-\mu^2)^{m/2}R^{m-2}\{\mu\, \partial\theta^*/\partial\mu - R\, \partial\theta^*/\partial R\}, \quad (6)$$

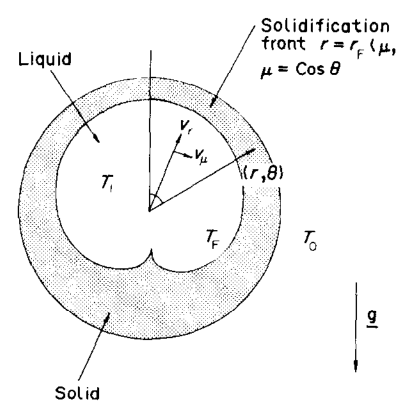

FIG. 1. Physical model and coordinates.

and

$$\nabla_R^2\theta^* - (1-\mu^2)^{(2-m)/2}R^{-m}\,\partial(\Psi^*,\theta^*)/\partial(R,\mu) = \partial\theta^*/\partial\tau^*. \quad (7)$$

Solid phase

The dimensionless temperature is given by

$$\nabla_R^2\theta = \partial\theta/\partial\tau. \quad (8)$$

Boundary conditions

In the liquid phase the initial conditions are

$$\theta^*(R,\mu,0) = 1, \quad \Psi^*(R,\mu,0) = 0 \text{ for } 0 \leqslant R \leqslant 1, \quad (9)$$

and for the solid phase

$$\theta(1,\mu,\tau) = 0 \text{ for } \tau > 0. \quad (10)$$

At the interface $R = R_F(\mu,\tau)$ for $\tau > 0$,

$$\theta^* = 0, \quad \theta = 1, \quad \Psi^* = \partial\Psi^*/\partial R = 0, \quad (11)$$

and the latent heat condition is

$$(\partial/\partial R - R^{-2}(1-\mu^2)\partial R_F/\partial\mu)(\theta-\theta^*) = \beta\,\partial R_F/\partial\tau. \quad (12)$$

Moreover at

$$\tau = 0, \quad R_F(\mu,\tau) = 1. \quad (13)$$

Initially the local structure of the solidification process on the surface of the sphere or cylinder will be that for a semi-infinite region at a temperature above the fusion temperature. The Neumann variables, see Carslaw and Jaeger [9], are introduced. These are:

$$\xi = (1-R)/2\tau^{1/2}, \quad \xi^* = (1-R)/2\tau^{*1/2},$$
$$E(\mu,\tau) = (1-R_F)/2\tau^{1/2} \quad (14)$$

for the solid and liquid regions and the interface location, respectively.

For small time assume regular perturbation expansions:

$$\theta(\xi,\mu,\tau) = \sum_{n=0}^{\infty} \theta_n(\xi,\mu)\tau^{n/2},$$
$$\theta^*(\xi^*,\mu,\tau) = \sum_{n=0}^{\infty} \theta_n^*(\xi^*,\mu)\tau^{*n/2}$$
$$E(\mu,\tau) = \sum_{n=0}^{\infty} E_n(\mu)\tau^{n/2} \quad (15)$$

and

$$\Psi^*(\xi^*,\mu,\tau^*) = \sum_{n=0}^{\infty} \Psi_{n+3}^*(\xi^*,\mu)\tau^{*(n+3)/2}.$$

Substitution of (15) into the governing equations (6)–(8) yields a system of ordinary differential equations for the perturbation functions $\theta_n(\xi,\mu)$, etc; the interface boundary conditions are derived using the method of Van Dyke [10]. These differential equations can be solved analytically, see Stead [11], but this approach proved to be limited and laborious.

On specifying γ, λ, P, Ra and β the three point boundary value equations for the thermal fields and stream function are integrated numerically using the Runge–Kutta–Merson technique (NAG library); the standard method of computing complementary and particular integrals coupled with an iterative method for evaluating the E_n was employed. Perturbation expansions were extended up to terms of fifth order. Convergence of expansions at dimensionless times such as $\tau = 0.8$ was established on comparing computed flow and heat transfer properties from truncated expansions at fourth and fifth order, respectively.

The position of the interface is given by the equation of the limacon of Pascal, namely

$$R = R_F(\mu,\tau) = 1 - E(\mu,\tau) = a(\tau) + \mu b(\tau), \quad (16)$$

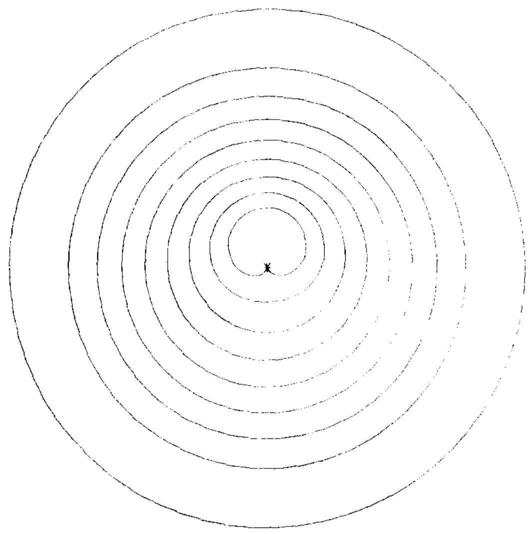

FIG. 2. Interface locations for the sphere at time intervals $\Delta\tau = 0.1$.

where

$$a(\tau) = 1 - 2\tau^{1/2}\sum_{n=0}^{5} E_n\tau^{n/2}, \quad b(\tau) = 2\tau^{5/2}(\bar{E}_4 + \bar{E}_5\tau^{1/2})$$

and

$$a(\tau) \leqslant b(\tau).$$

The volume of liquid remaining is

$$V(\tau) = 2\pi^{m-1}a^{m+1} \times \int_{-1}^{+1}\int_{R=0}^{R=1-E(\mu,\tau)} R^m(1-\mu^2)^{(m-2)/2}\,\mathrm{d}R\,\mathrm{d}\mu. \quad (17)$$

3. RESULTS AND DISCUSSION

The data to be used as input for this study is available for metal alloy systems and thermal storage fluids, see refs. [8] and

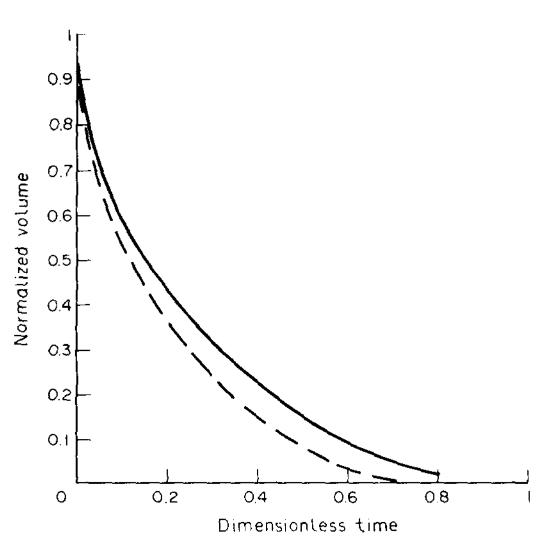

FIG. 3. Normalised volume of liquid as a function of the time.

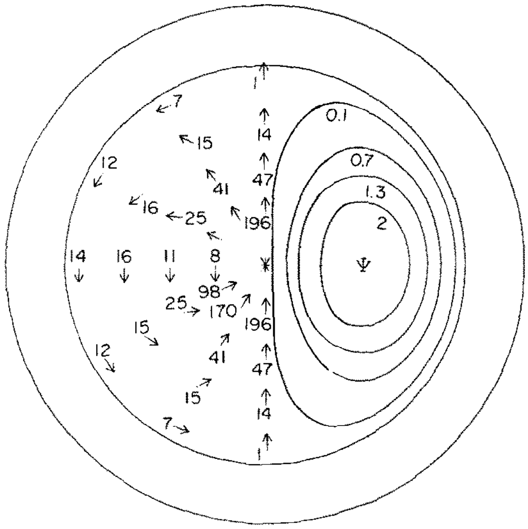

FIG. 4. Dimensionless stream function and velocity at $\tau = 0.1$.

[3]. We take the latter fluid for illustration using the values $\gamma = \lambda = 0.1, \beta = 3, P = 13.4$ and $Ra = 8 \times 10^3$. Results for the sphere are discussed; those for the cylinder are almost identical in structure.

The axisymmetric convective flow rises in the form of a jet along the axis $\theta = \pi$ and $\theta = 0$ to form a forward stagnation point at the north pole $\theta = 0$. It returns downward on both sides of the axis bathing the interface and collides to give a backward stagnation point at $\theta = \pi$. As the liquid volume decreases and looses its sensible heat the density differences increase leading to increased convection. In the early part of the process this form of thermal spin-up initiates the formation of the cusp at the south pole. At the dimensionless time $\tau = 0.4$ the liquid core temperature is below the initial temperature. Thus as the process evolves the liquid temperature will approach the fusion temperature and the circulation will cease.

Interface locations are displayed in Fig. 2 at intervals of $\Delta\tau = 0.1$. The volume of liquid as a function of time is given in Fig. 3; here the dashed curve gives the volume remaining when convection is ignored and such results were found to be in good agreement with the analytical studies of Stewartson and Waechter [1]. Figure 4 displays the streamlines and velocity av/k at $\tau = 0.1$. Detailed results on the temperature distribution and heat transfer coefficients (at the interface and sphere surface) have also been evaluated.

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Analytical solution for the buoyancy flow during the melting of a vertical semi-infinite region

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NOMENCLATURE

g	acceleration of gravity
k	thermal conductivity
p	pressure
Pr	Prandtl number of the melt, ν/α_1
Ste	Stefan number $k_1(T_0 - T_i)/\rho\alpha_1 L_m$, where L_m = latent heat
T_{avg}	$\frac{1}{2}(T_0 + T_m)$
v	velocity of the melt in the y direction.

Greek symbols

α	thermal diffusivity
β	coefficient of thermal expansion, assumed constant
ν	kinematic viscosity of the melt
ρ	density at T_{avg}.

Subscripts

m	at the melting front, $x = X(t)$
1	liquid phase
2	solid phase
21	solid phase divided by liquid phase.

1. INTRODUCTION

THE IMPORTANCE of natural convection in the Stefan's problem received attention only in recent years. Both experiments [1] and numerical analyses [2] indicate that the buoyancy-driven